# Research statement

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## 1 Introduction and overview

My main research interests lie in stable homotopy theory. The field is a great source of invariants of spaces as well as examples of algebraic objects. From this perspective, my broad interests lie in concrete computations that shed light on geometric questions by interpreting these computations (see section 3 for example). I am specifically interested in highly structured ring spectra and the residue that structure leaves behind on the level of more discrete invariants such as homotopy groups. This structure can frequently be used to enhance computational techniques (see section 2) as well as give algebro-geometric interpretations (see section 3). One avenue for interpreting computations is that of derived or homotopical algebraic geometry. It provides a way of understanding $E_\infty$-ring spectra as geometric objects. I believe that the intuition from algebraic geometry that is imported in this way can shed a great deal of light on the stable homotopy category. An example of this is the reinterpretation of the Hopkins-Neeman thick subcategory theorem as a computation of the closed points of $Spec(S^0)$, see [3].

Another aspect of my research is power operations and multiplicative structures of spectral sequences. This extra structure can help significantly in the determination of differentials. It also helps to shed light on the nature of $E_\infty$-ring structures by providing for methods of computation. My focus has been on extending classical results in the Adams spectral sequence for $H_\infty$-ring spectra to the $C_2$-equivariant Adams spectral sequence. The hope of this project was to give a more general argument establishing the differential

$$d_2(h_5) = (h_0 + \rho h_1)h_4^2$$

in the motivic Adams spectral sequence over $Spec(\mathbb{R})$. This is one of the motivic Hopf invariant one differentials that Isaksen has arrived at through ad hoc methods. If Bruner's methods could be brought to bear on this differential it would very likely be able to determine many of the higher Hopf invariant one differentials that are not currently known. The formulas that we have for the $C_2$-equivariant Adams spectral sequence do not imply this differential as $h_4$ is not a permanent cycle. The complete formula that I have proved as well as some elaborations and future directions can be found in section 2.

Both of these parts of my research have the underlying theme of doing computations by taking into account more structure. This is the philosophy that motivates my work. I am interested in understanding multiplicative structures in the newer fields of $G$-equivariant and motivic homotopy theory. This aspect of those fields is now growing and there are more tools for computations so that we can understand how they relate to the structures we understand classically.

## 2 Adams spectral sequences

In his work on the Adams spectral sequence, Bruner gives a formula relating differentials and power operations on the $E_2$-page see Chapter 6 section 1 of [4]. The following is a result I have proved for permanent cycles.

**Theorem 1** (Tilson)**.** *In the $C_2$-equivariant Adams spectral sequence converging to the $C_2$-equivariant stable homotopy groups of spheres, for a permanent cycle $x \in \mathrm{Ext}_{\mathcal{A}}^{s,t,q}(\mathbb{M}_2, \mathbb{M}_2)$, we have*

$$d_2 Sq^{i-1}x = \alpha_{i,q} Sq^i x$$

*where*

$$\alpha_{i,q} = \begin{cases} h_0 + \rho h_1 & i \equiv 1, q \equiv 0 \pmod 2 \\ h_0 & i \equiv 1, q \equiv 1 \pmod 2 \\ \rho h_1 & i \equiv 0, q \equiv 1 \pmod 2 \\ 0 & i \equiv 0, q \equiv 0 \pmod 2 \end{cases}$$

*in* $\mathrm{Ext}^{1,1,0}_{\mathcal{A}}(\mathbb{M}_2, \mathbb{M}_2)$.

Here, $\mathbb{M}_2 = H^{**}(*) =$ where $\mathbb{M}_2 := \mathbb{F}_2[\tau, \rho] \oplus N$ and $N$ is a graded $\mathbb{F}_2[\tau, \rho]$-module. The $\mathbb{F}_2[\tau, \rho]$ module $N$ is $\mathbb{F}_2 \frac{\theta}{\tau^i \cdot \rho^j}$ with $\theta \in N_{0,-2}$ annihilated by both $\rho$ and $\tau$, [5] by Caruso.

It is worth pointing out that this theorem applies to the $C_2$-equivariant Adams spectral sequence that is only concerned with the $RO(C_2)$-grading and not the Mackey functor structure. This apparent choice is motivated by its compatibility with the motivic Adams spectral sequence over $Spec(\mathbb{R})$ of Dugger and Isaksen, developed in [6]. Another reason for this choice is that much of the necessary input is unavailable, such as the Steenrod algebra for the equivariant cohomology taking values in Mackey functors being unknown.

This project has three different follow up projects:

1. Prove analogous results for the motivic Adams spectral sequence.
2. Extend the result to non-permanent cycles.
3. Work in a different equivariant context such as changing the group or the type of cohomology (for example, working with cohomology that takes coefficients in the Burnside Mackey functor).

The first of these requires proving a motivic analog of the following result.

**Theorem 2** (Tilson)**.** $\widehat{E}\Sigma_{2+}^{(n,m)} \wedge_{\Sigma_2} (S^{p,q} \wedge S^{p,q}) \cong \Sigma^{p,q}\mathbb{R}\mathrm{P}^{n+p,m+q}/\mathbb{R}\mathrm{P}^{p-1,q}$

The left hand side is the $(n, m)$-skeleton of the extended power of $S^{(p,q)}$ as $\widehat{E}\Sigma_2^{(n,m)}$ is

$$S^{n,m} \cong \mathbb{R}^{n-m}_{triv} \oplus \mathbb{R}^m_\sigma$$

(one point compactification of $n - m$ copies of the trivial representation direct sum $m$ copies of the sign representation). The group $\Sigma_2$ acts on $S^{n,m}$ via the antipodal map. The main difficulty with constructing differentials in the motivic Adams spectral sequence is precisely this result. It is not clear what the analog of projective spaces should be. At first glance, the suggestion of using Voevodsky's $B\mu_2$ seems obvious. However, using a skeletal filtration of $E\mu_2$ instead of the "constant" motivic space $E\Sigma_2$ may give rise to a different algebra of operations on the $E_2$-page of the spectral sequence. There is room for both perspectives though, and in fact it should be the case that each type of norm construction should provide its own family of differentials.

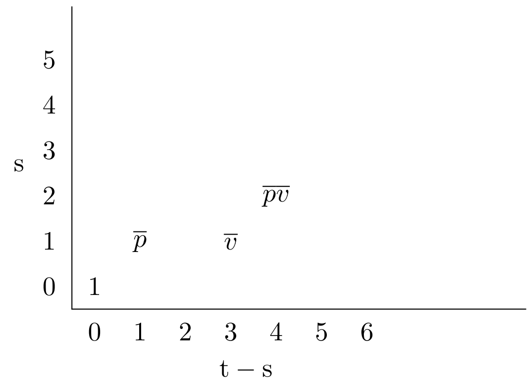
Extending this work to other equivariant Adams spectral sequences, such as those taking into account the Mackey functor structure are also very interesting. However, there are roadblocks as mentioned above. Flatness of the relevant Steenrod algebra over the coefficients is a potential problem. The state of the art on equivariant Steenrod algebras is also not where it needs to be. A natural direction to move in is then that of slice spectral sequences. This should be approachable given the work on the multiplicativity of the slice tower by Gutiérrez-Röndigs-Spitzweck-Østvær in [9]. It would also require more information about motivic $E_\infty$-structures.

# 3 Relative smash products

In the first part of my thesis, see [15], I worked on power operations and the multiplicativity of the Künneth spectral sequence (which I will abbreviate as KSS)

$$\mathrm{Tor}_p^{R_*}(A_*, B_*)_q \Rightarrow \pi_{p+q}(A \wedge_R B)$$

where $A$ and $B$ are commutative $R$-algebras for $R$ a commutative $S$-algebra in the sense of EKMM ([8]). In [2], they give an incorrect proof that this spectral sequence is multiplicative. I show that not only is the spectral sequence multiplicative but it has a natural $H_\infty$-structure in a filtered sense. Unlike the situation with the Adams spectral sequence, this structure is not detectable on the $E_2$-page of the spectral sequence (as a corollary of Tate's work in [14]). Despite this, the naturality of the $H_\infty$-structure I construct allows one to push power operations from one relative smash product to another. These power operations paired with the interpretation of elements in $\mathrm{Tor}_1$ as differences of null-homotopies leads to interesting results. For example, the KSS converging to $\pi_*(\mathrm{H}\mathbb{F}_p \wedge_{ku} \mathrm{H}\mathbb{F}_p)$ has the following particularly simple form.




Here, $\overline{p}$ and $\overline{v}$ can each be realized as universal differences of null homotopies of $p \in \pi_0(ku)$ and $v \in \pi_2(ku)$ respectively. This means that for any two commutative $\mathrm{H}\mathbb{F}_p$-algebra structures on $X$, say $f$ and $g$, that restrict to the same commutative $ku$-algebra structure we get an element $d(f,g)(\overline{p}) \in \pi_1(X)$ that vanishes if and only if $f$ and $g$ cone off $p$ in homotopic ways, see Lemma 6.6 of [15]. The same can be said of $\overline{v}$ and any other class on the 1-line of KSS constructed in a certain way, see Proposition 6.5 of [15]. There is a power operation $Q^2(\overline{2}) = \overline{v}$ in $\pi_*(\mathrm{H}\mathbb{F}_2 \wedge_{ku} \mathrm{H}\mathbb{F}_2)$. This implies that if one constructs a commutative $\mathrm{H}\mathbb{F}_2$-algebra from a commutative $ku$-algebra by coning off 2 then the choices for how to cone off $v$ are constrained. At an odd prime $p$, we do not have such an operation for $ku$. However, we do have $Q^1(\overline{p}) = \pm\overline{v_1}$ in $\pi_{2(p-1)}(\mathrm{H}\mathbb{F}_p \wedge_\ell \mathrm{H}\mathbb{F}_p)$ where $\ell$ is the connective Adams summand.

Similar computations can be done of the action of the Dyer-Lashof algebra on $\mathrm{H}\mathbb{F}_p \wedge_{MU} \mathrm{H}\mathbb{F}_p$. They easily imply the following result.

**Theorem 3** (Tilson)**.** *Let $I$ be an ideal of $MU_*$ generated by a regular sequence. If $I$ contains a finite nonzero number of the $x_{p^i-1}$, then the quotient map $MU \to MU/I$ cannot be realized as a map of commutative $S$-algebras.*

The result is proved by considering the effect of such a map on the homotopy of relative smash products as algebras over the Dyer-Lashof algebra.

These results have two natural directions emanating from them:

1. Can the action of the $E$-theoretic Dyer-Lashof algebra be computed on $E \wedge_{MU} E$ to give similar results as the above when $E$ is of a different chromatic type than $\mathrm{H}\mathbb{F}_p$?

2. Why does the relative smash product $\mathrm{H}\mathbb{F}_p \wedge_R \mathrm{H}\mathbb{F}_p$ as a commutative $\mathrm{H}\mathbb{F}_p$-algebra know so much about commutative $R$-algebras?

The first question is not so easy. The $MU_*$-module structure on $E_*$, in order to obtain results about $E_\infty$-structures, must be induced by a map of $E_\infty$-ring spectra. Such $E_\infty$-complex orientations are not very well understood (although there has been recent progress [10]). Further, any such $E$ that is Landweber exact will not be amenable to our approach used for $\mathrm{H}\mathbb{F}_p$ as Laures[1] has shown that such modules are flat over $MU_*$. Our approach there was to consider the spectral sequence converging to $\mathrm{H}(\mathrm{H}\mathbb{F}_p; \mathbb{F}_p)$ and push

[1] private communication

the action of the Dyer-Lashof algebra along this map of spectral sequences. However, the above mentioned flatness implies that the KSS collapses on the 0-line whereas all of the interesting classes live on the 1-line. This is one of several obstacles.

The second question is part of current work in progress. This work concerns constructing a formalism so that these relative smash products can be interpreted as objects in derived (or homotopical) algebraic geometry. The current form of the project involves an association of a derived affine group scheme (in the sense of [13]) to a map

$$\varphi : R \longrightarrow k$$

of commutative $k$-algebras. This group scheme is a proposed definition for the fundamental group of $Spec(R)$ based at $Spec(k)$. The definition is relatively straightforward, but it will only contain very local data about $Spec(R)$ at the point $\varphi$. The relative smash product $k \wedge_R k$ generates the whole fundamental group, and in this way we see why the relative smash product "knows about" $Spec(R)$. Further aspects of this involve understanding the notion of a fundamental groupoid in this setting as well as moving away from the restriction of having a map of $k$-algebras. This last restriction is undesirable because of various objects of interest are not "pointed schemes" in this sense. For example, the computations above are examples of such objects. This is something I have been discussing with Jon Beardsley and Markus Spitzweck.

Another avenue is to perform more computations. Lukas Katthän and I have computed the $E_2$ page of the KSS converging to $\mathrm{H}\mathbb{F}_p \wedge_{BP\langle n\rangle} e_n$ where $BP\langle n\rangle$ is the truncated Brown-Peterson spectrum and $e_n$ is a connective cover of Morava $E$-theory. The spectral sequence based on $BP\langle n\rangle$ is not multiplicative as it is currently unknown if $BP\langle n\rangle$ is an $E_\infty$-ring spectrum for $n$ larger than 2. The product structure on the $E_2$-page is a bit involved in general, but for $p = n = 2$ we would have that

$$\pi_*(\mathrm{H}\mathbb{F}_2 \wedge_{BP\langle 2\rangle} e_2) \cong \mathbb{F}_4[u_1, u, f_{12}]/(u^3, uu_1, uf_{12}, f_{12}^2)$$

with $u$ and $u_1$ coming from $\pi_2(e_2)$ and $\pi_0(e_2)$ respectively and $f_{12}$ is in degree 7, assuming that the spectral sequence is multiplicative. I have an approach that bases the spectral sequence on $BP$ which is an $E_4$-ring spectrum but have not yet finished the details. If completed, this work would easily imply an injection

$$\mathrm{H}_*(BP\langle 2\rangle; \mathbb{F}_p) \longrightarrow \mathrm{H}(e_n; \mathbb{F}_p)$$

which is a question raised by Andy Baker.

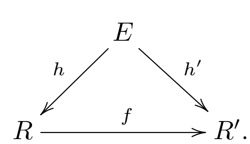
# 4 Moduli of $A_\infty$-structures

In work with John Klein, we found a fiber sequence that explains the role of $THH$-cohomology in the study of $A_\infty$-structures on both spaces and ring spectra. This project was motivated by a question of John Klein's as to why $THH$-cohomology shows up in the work of Robinson and Angeltveit on $A_\infty$-structures, see [12] and [1] respectively. A main observation of John Klein's was that the obstruction theory of Robinson involves an explicit model for an $A_\infty$-operad and so that it was not a "coordinate free" approach. We were able to resolve this point by constructing a fiber sequence that involves both a form of $THH$-cohomology and the moduli space of $A_\infty$-structures.

The moduli space $\mathcal{M}_E$ of $A_\infty$-structures on a given $k$-module $E$ is the classifying space of the category with objects associative $k$-algebra $R$ equipped with a weak equivalence

$$h : E \longrightarrow R$$

of $k$-modules. A morphism in this category is a map $f$ of $k$-algebras making the following diagram of $k$-modules commute



In order to locate this moduli space in a fiber sequence we use the topological Hochschild cohomology space of a $k$-algebra $R$ with coefficients in a bimodule $M$ is given by

$$\mathrm{HH}^\bullet(R;M) := \underline{Hom}_{R^e}(R,M).$$

Here, $\underline{Hom}_{R^e}$ means a derived mapping space of maps of $R$-bimodules. We also denote by $T$ the functor that takes a $k$-module to the free $k$-algebra generated by that module. With these definitions we now have the following theorem.

**Theorem 4** (Klein-Tilson)**.** *When $R \in k-Mod$ is equipped with an algebra structure there is a homotopy fiber sequence*

$$\Omega^2\mathcal{M}_R \to \mathrm{HH}^\bullet(R;R) \to \mathrm{HH}^\bullet(TR;R)\,,$$

*in which $\Omega^2\mathcal{M}_R$ is identified with the homotopy fiber at the basepoint of $\mathrm{HH}^\bullet(TR;R)$ that is associated with the $TR$-bimodule map $TR \to R$.*

In fact, this fiber sequence admits a double delooping so that we have the fiber sequence

$$\mathcal{M}_R \to B^2\mathrm{HH}^\bullet(R;R) \to B^2\mathrm{HH}^\bullet(TR;R)\,.$$

There is also an unstable version of this result in that paper which addresses the moduli of loop space structures on a given loop space. One possible future direction would be to investigate the existence of such fiber sequences relating the moduli of $E_n$-algebra structures to higher $THH$-spaces, or even $\mathcal{O}$-algebra structures to the Quillen homology of Rezk. In the case of $E_n$-algebras there is a major obstacle. At a key point in the proof we need that the equivalence

$$F(S^1_+, R) \simeq R \wedge \Sigma^{-1}R$$

is an equivalence of associative algebras where the right hand side is the trivial square zero extension. This is claimed by Lazarev in [11], but his proof works for all $E_n$-algebras which is well known (as Lazarev himself remarks after his proof) to be false. As this is not a common condition for $\mathcal{O}$-algebras it is not so likely that the above fiber sequence can be extended to other operads.

# 5 Current and future projects

I have several ideas for future projects. Each of these projects has already been preliminarily discussed with the mentioned parties.

One current project is on the determination of the Dyer-Lashof algebra for $p$-complete connective complex $K$-theory. The Dyer-Lashof algebra for periodic $K$-theory is known by work of McClure, see [4], but it seems not much is known about the connective situation. This is complicated by the interaction between ordinary group homology and connective $K$-theory. Even for the computation of $ku_{p*}(B\Sigma_2)$ we see that the inversion of the Bott element and $p$-completion organizes an infinite amount of torsion into a single copy of the $p$-adics; this does not happen in the connective situation. Martin Frankland and I have begun working on this. We expect that much can be learned as the representation theory of $\Sigma_p$ for $p$ a prime is relatively well understood and both the $\mathrm{H}\mathbb{F}_p$ and $KU_p$ Dyer-Lashof algebras have all of their primary operations determined by the homology of $B\Sigma_p$ we expect this to be the case for $ku_p$ as well.

One future project regards operations on categories of modules over $E_n$-algebras. There is a result of Mandell that establishes what structure one has on the derived category of an $E_n$-ring spectrum when $n \in \{1,2,3,4\}$. In particular, when $R$ is an $E_4$-ring spectrum the homotopy category of $Mod_R$ has a symmetric monoidal structure. It is also the case that $E_n$-algebras in chain complexes have bracket operations defined on their homology that vanish if the $E_n$-algebra structure extends to an $E_{n+1}$-algebra structure. An interesting project would be to lift this operation to something defined on the homotopy category of modules so that its non vanishing can be seen as a concrete obstruction to the symmetry isomorphism required of a symmetric monoidal category. I have had preliminary discussions with Rune Haugseng and Geoffroy Horel.

Rune Haugseng currently has a related project that would allow is to identify an $E_n$-monoidal $dg$-category with algebras over a certain $\infty$-operad. This would allows us to compute the operations that act on the graded homotopy category of an $E_n$-monoidal $dg$-category.

Another future project is to produce more computations of relative smash products and study operations on them. Bousfield and Dwyer were able to construct higher divided power operations on the homotopy of simplicial commutative $\mathbb{F}_p$-algebras (see [7] and Bousfield's unpublished notes "Homogeneous functors and derived functors" as well as "Operations on derived functors of nonadditive functors"). The $E_2$-pages of many KSSs possess such a structure but it is unclear what structure on the homotopy groups of spectra they might be detecting. There are also many more computations of relative smash products to be done. Once there is a proper interpretation of $A \wedge_R B$ as a derived algebro-geometric object (ideally part of the data of a fundamental groupoid) then these computations will have a natural home. I have spoken to Clark Barwick about this as well as Jon Beardsley. Clark Barwick's interest in this was motivated by understanding the derived crystalline site. My main motivation is that there is a multiplicative structure on ring spectra that we do not yet know about.

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