Research Statement

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1 Introduction

I work in the field of stable homotopy theory. My work is focused on using power operations in conjunction with spectral sequences to learn about the stable homotopy category. In the 1990's models of spectra were developed that enabled us to treat the category of spectra as a category of modules over the sphere spectrum S. This major advance has made new mathematics possible as well as more conceptual.

I am interested in computing invariants of ring spectra and using them to better understand the structure of the stable homotopy category. The first part of my thesis was concerned with power operations in the Kunneth spectral sequence. I was able to compute the action of these operations on relative smash products of Eilenberg-MacLane spectra. I was able to interpret these computations in terms of independent R algebra structures on algebras over an Eilenberg-MacLane spectrum. The second part of my thesis is concerned with power operations in the C_2-equivariant Adams spectral sequence and their use to compute differentials. We can use the category of C_2-equivariant spectra to approximate motivic spectra over $Spec(\mathbb{R})$ because of the realization map from motivic spectra over $Spec(\mathbb{R})$ to C_2-equivariant spectra. C_2-equivariant spectra are useful for this purpose because their homotopy theory is more familiar.

2 Power Operations

Power operations can be thought of as evidence of a non-trivial E_∞ structure, i.e. as obstructions to commutativity in the strictest sense. These operations were first constructed by Araki and Kudo for singular homology of infinite loop spaces with coefficients in $\mathbb{F}_2$. Later, Steinberger computed the action of such operations on the ordinary homology of E_∞ ring spectra. He was able to apply these computations to deduce splittings of interesting spectra. For example, Steinberger's computations imply that the Thom spectrum MO splits as a wedge of Eilenberg-MacLane spectra, but not necessarily as a ring. See [**?**] for many helpful explanations of power operations, where the homology theories that are specifically considered are singular homology, stable homotopy, and complex K-theory.

The origin of these operations is as follows. The product map of a commutative ring spectrum factors through the extended power (or homotopy orbit construction)

$$\begin{array}{ccc} R^{\wedge r} & \longrightarrow & R \\ & \searrow \quad \nearrow & \\ & E\Sigma_{r+} \wedge_{\Sigma_r} R^{\wedge r} & \end{array}$$

Let $D_r X$ denote $E\Sigma_{r+} \wedge_{\Sigma_r} X^{\wedge r}$. If Y and X are both E_∞ ring spectra we can construct homology operations in $Y_* X$. Given a homology class $x \in Y_k X$ and $\alpha \in Y_n D_r S^k$ we can define $\alpha_*(x) \in Y_n X$ as follows. We represent each homology class as a map:

$$S^k \xrightarrow{x} Y \wedge X \quad \text{and} \quad S^n \xrightarrow{\alpha} Y \wedge D_r(S^k)$$

Then $\alpha_*(x)$ is the composition:

$$S^n \xrightarrow{\alpha} Y \wedge D_r(S^k) \xrightarrow{1_Y \wedge D_r(x)} Y \wedge D_r(Y \wedge X) \longrightarrow Y \wedge Y \wedge X \xrightarrow{\mu_Y \wedge 1_X} Y \wedge X$$

More details can be found in [?], see for example sections IV.7 or IX.1. Maps of E_∞ ring spectra must preserve these operations, and this criterion is useful in computations.

3 Power Operations in Spectral Sequences

Throughout this section we will let H denote the mod 2 Eilenberg-MacLane spectrum. It is a commutative ring spectrum and $\pi_* H \cong \mathbb{F}_2$ concentrated in degree 0. The Kunneth spectral sequence for a right R-module A and a left R-module B, is

$$Tor_p^{\pi_* R}(\pi_* A, \pi_* B)_q \Rightarrow \pi_{p+q}(A \wedge_R B)$$

The E_2 term of the spectral sequence is a ring when R is a commutative S-algebra and A and B are both commutative R-algebras. When this is the case the Kunneth spectral sequence is multiplicative: the product at E_2 converges to the product on the target. This follows easily if one constructs the spectral sequence by filtrations which are preserved by the product map. A key step is to establish an equivalence of categories between resolutions by free R-modules and filtrations whose filtration quotients are free R-modules and which have a certain exactness property. While there does not seem to be a general criterion for showing that a spectral sequence has power operations, these constructions not only show that the Kunneth spectral sequence is multiplicative, but that it also has power operations. The following comparison theorem accomplishes these goals.

Theorem 1. *Given a map $f : Y \to A$, a free resolution $F_\bullet \to A$, and a free filtration of $Y_\bullet \subset Y$ there is a map of filtrations $Y_i \xrightarrow{f_i} A_i$ where $A_\bullet \subset A$ is the filtration associated to the free resolution $F_\bullet \to A$.*

The product structure on the Kunneth spectral sequence follows from taking $f : Y = A \wedge A \to A$ to be the product map and $Y_n = \bigcup_{i+j=n} A_i \wedge A_j$. This corrects the incorrect proof given in [?]. The construction of power operations comes from taking $f_r : Y = D_r A \to A$ to be the E_∞ structure maps and $Y_n = \bigcup_{k+\Sigma_{i=1}^r l_i = n} E\Sigma_{r+}^{(k)} \wedge_{\Sigma_r} \bigwedge_i A_{l_i}$. Having established these general facts, we use the Kunneth spectral sequence to compute operations on some relative smash products. The theorem above is given by natural constructions so that the operations are preserved by maps of commutative ring spectra. We can use this naturality and an early computation of Steinberger to compute the operations we are interested in.

Theorem 2. *(Steinberger 1977) $H\mathbb{F}_{2*}H\mathbb{F}_2 \cong \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$ is generated as an algebra over the Dyer-Lashof/Araki-Kudo algebra by the operations $Q^{2^i-2}(\xi_1) = \tau_*(\xi_i)$ where $\tau : H\mathbb{F}_{2*}H\mathbb{F}_2 \to H\mathbb{F}_{2*}H\mathbb{F}_2$ is the conjugation induced by the twist map.*

Using this result and the action map $H \wedge R \to H$ we can compute invariants of E_∞ structures. The technique works by mapping a Kunneth spectral sequence that computes $H_* H$ to one computing $\pi_* H \wedge_R H$. The action map $H \wedge R \to H$ induces the following map of spectral sequences:

$$\begin{array}{ccc} Tor_p^{\pi_* R}(H_* R, \pi_* H)_q & \Longrightarrow & \pi_{p+q}(H \wedge R \wedge_R H) \cong H_{p+q} H \\ \downarrow & & \downarrow \\ Tor_p^{\pi_* R}(\pi_* H, \pi_* H)_q & \Longrightarrow & \pi_{p+q}(H \wedge_R H) \cong H_{p+q}^R H \end{array}$$

This leads to the following interesting computations.

Corollary 3. *Let ku denote the connective K-theory spectrum and $BP\langle 2\rangle$ denote the E_∞ model of of the truncated Brown Peterson spectrum, recently constructed by Lawson and Naumann in [?]. Then $H_*^{ku}H$ is an exterior algebra $E_{\mathbb{F}_2}[\overline{2}, \overline{v}]$ ove $\mathbb{F}_2$ with $Q^2(\overline{2}) = \overline{v}$ and $H_*^{BP<2>}H$ is an exterior algebra $E_{\mathbb{F}_2}[\overline{2}, \overline{v_1}, \overline{v_2}]$ over $\mathbb{F}_2$ generated by $Q^2(\overline{2}) = \overline{v_1}$ and $Q^6(\overline{2}) = \overline{v_2}$.*

4 C_2-Equivariant Adams Spectral Sequences

The category of C_2-equivariant spectra is the target of a realization map from motivic spectra over $Spec(\mathbb{R})$, as ordinary stable homotopy theory is the target of a realization map from motivic spectra over $Spec(\mathbb{C})$. Bruner developed, in his thesis [?], a way of relating algebraic power operations in the Adams spectral sequence to differentials in that spectral sequence. Using his techniques, I produce new differentials through the use of power operations. Since classes in the Adams spectral sequence can be represented by maps of spectra, we can use extended power constructions to compute differentials using the cell structure of those extended powers. Most of Bruner's constructions can be carried out in the C_2-equivariant setting.

The formalism developed in [?] and the sequel [?] enable us to show that an Adams tower for a C_2-equivariant commutative ring spectrum has an extended power structure. Using this structure and the cell structure of equivariant truncated projective spaces we can compute some differentials. While definitive results akin to [?] await a better understanding of the cohomology of some C_2-equivariant (or motivic) spectra, we already have new differentials. We define $D_2X = S_+^{\infty,0} \wedge_{C_2} X \wedge X$ where $S^{\infty,0}$ is the unit sphere of the trivial universe and C_2 acts on $X \wedge X$ by swapping the factors. Every E_∞-ring spectrum in the C_2-equivariant setting has a structure map $D_2X \to X$. Our choice of the "naive" extended power is influenced by the structure we see on the algebraic level, the cobar complex that computes the E_2 term of the Adams spectral sequence. As of now, the most concrete result, and an initial motivation for the project, are the following propositions:

Theorem 4. *In the C_2-equivariant Adams spectral sequence converging to the stable homotopy groups of spheres, for $x \in Ext_{\mathcal{A}}^{s,t,u}(\mathbb{M}_2, \mathbb{M}_2)$ we have*

$$d_2 Sq^{i-1}x = \alpha_{j,u} Sq^i x$$

where $j = t - s + i$

$$\alpha_{j,u} = \begin{cases} h_0 + \rho h_1 & j \equiv 1, u \equiv 0 \pmod 2 \\ h_0 & j \equiv 1, u \equiv 1 \pmod 2 \\ \rho h_1 & j \equiv 0, u \equiv 1 \pmod 2 \\ 0 & j \equiv 0, u \equiv 0 \pmod 2 \end{cases}$$

in $Ext_{\mathcal{A}}^{1,1,0}(\mathbb{M}_2, \mathbb{M}_2)$.

This gives the following corollary

Proposition 5. $d_2(h_5) = (h_0 + \rho h_1)h_4^2$ *where $h_i \in Ext_{\mathcal{A}}^{1,2^i,2^{i-1}}(\mathbb{M}_2, \mathbb{M}_2)$ and $\rho \in Ext_{\mathcal{A}}^{0,-1,-1}(\mathbb{M}_2, \mathbb{M}_2)$*

This is the first differential we were hoping to explain. Classically, the differential is $d_2(h_5) = h_0 h_4^2$. This is consistent with the fact that the realization map sends ρ to 0. In general, $d_2 h_{i+1} = (h_0 + \rho h_1)h_i^2$, and $i = 4$ is the lowest degree in which the "exotic" $\rho h_1 h_i^2$ term is non-zero. The general form of the result for computing differentials in the spectral sequence is work in progress. It will be very interesting to see the equivariant picture differs from the classical one in the higher differentials.

5 Future Work

I would like to extend my computations for algebras over Eilenberg-MacLane spectra to include $H \wedge_{e_n} H$ where e_n is a connective version of Morava E-theory. The homology of these connective theories is not explicitly mentioned in the literature, so those computations will be necessary first. Another project in this direction is to compute the action of the relevant Dyer-Lashof algebra on relative smash products of p-complete K-theory and Morava E theory spectra of height 2 over using work of McClure [?] and Rezk [?] respectively, i.e. $K_p \wedge_{MU} K_p$ and $E_2 \wedge_{MU} E_2$. Recent correspondence with Szymik has suggested working on relative smash products of Eilenberg-MacLane spectra over Postnikov sections of the sphere spectrum, which is currently in progress. One direction for future work is in better understanding applications of my computations of relative smash products. I want to explore the relationship between different types of moduli spaces, $\mathfrak{M}^R_X(Y)$, of E_∞-X algebra structures on Y that restrict to a given E_∞-R-algebra structure on Y and the spaces $E^R_\infty(X^{\wedge_R n}, Y)$.

Other potential projects are attempting to work with the Motivic Adams spectral sequence more directly. This requires that we realize the extended power constructions as more familiar objects in Motivic homotopy theory whose cell structure we can try to understand. The goal of the work on the C_2-equivariant Adams spectral sequence was to try to approximate the Motivic homotopy theory over $Spec(\mathbb{R})$. It could be very advantageous to understand the motivic picture becasue the relevant Ext and cohomology groups are easier to work with.

Another project seemingly unrelated to my previous work is to attempt to understand tensor triangulated geometry, see [**?**], through a functor of points approach and to try and see what this can tell us about the category of commutative ring spectra. Balmer has already used the Classification of Thick Subcategories result due to Devinatz, Hopkins, and Smith to compute the spectrum of the stable homotopy category of finite complexes. I would like to understand how this could be used to understand ring spectra through the formalism of Higher Algebraic Geometry as developed by Toen and Vessozi, see [**?**].

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